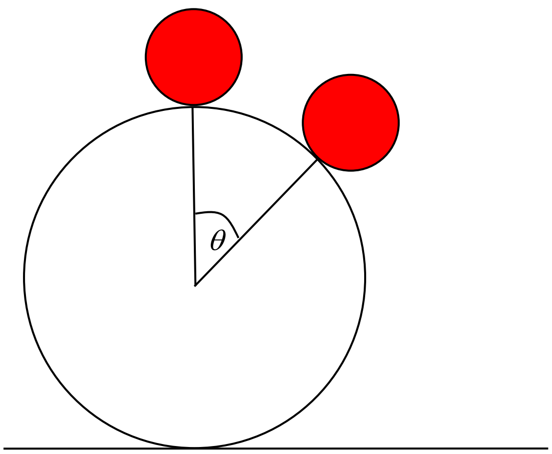
Teacher notes
Topic A

Rolling ball falling off a sphere

In a previous note we examined the problem of a ball (actually treated as a point particle) that was placed at the top of a sphere of radius R and given the slightest push so that it moves away. At what angle θ does the ball lose contact with the sphere? We now examine the same problem treating the ball as a real ball with moment of inertia that rolls without slipping on the big sphere. The radius of the ball is r.

Do this after first reviewing the case of the point particle.



The center of mass of the sphere will fall a vertical distance $R+r-(R+r)\cos\theta=(R+r)(1-\cos\theta)$ and so the speed at the point where the ball will leave the sphere is found by energy conservation to be:

$$0+mg(R+r)(1-\cos\theta)=\frac{1}{2}mv^2+\frac{2}{5}mr^2\omega^2+0$$

We assume rolling without slipping so $\omega=\frac{v}{r}$ and so

$$mg(R+r)(1-\cos\theta)=\frac{1}{2}mv^2+\frac{1}{2}\times\frac{2}{5}mr^2\frac{v^2}{r^2}$$

$$2g(R+r)(1-\cos\theta)=v^2+\frac{2}{5}v^2=\frac{7}{5}v^2$$

$$v^2=\frac{10}{7}g(R+r)(1-\cos\theta)$$

The rest of the analysis is the same as in the point particle case. We want the normal force to go to zero and so

$$mg\cos\theta = m\frac{v^2}{R+r}$$
$$= \frac{10}{7}mg(1-\cos\theta)$$

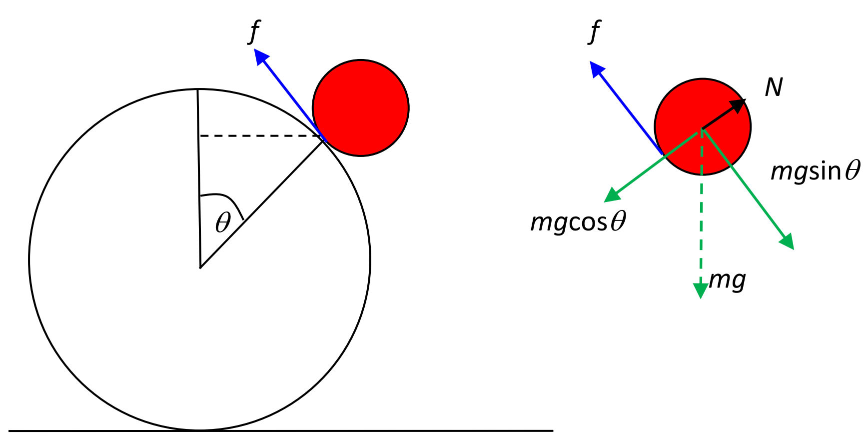
This gives:

$$7\cos\theta = 10(1-\cos\theta) \Rightarrow \cos\theta = \frac{10}{17}.$$

Hence $\theta \approx 54^\circ$. The point particle case gave $\theta \approx 48^\circ$.

Notice that the answer is independent of both *R* and *r*.

Notice that the normal force and the weight of the ball do not produce any torque about a horizontal axis through the center of mass of the ball. So, if the ball is to rotate, we need a frictional force acting as shown in the figure below.



The torque is fr and so

$$fr = I\alpha$$
$$= \frac{2}{5}mr^2\frac{a}{r}$$
$$f = \frac{2}{5}ma$$

From Newton's second law:

$$mg\sin\theta - f = ma$$
$$mg\sin\theta - \frac{2}{5}ma = ma$$
$$\frac{7}{5}ma = mg\sin\theta$$
$$a = \frac{5g\sin\theta}{7}$$

This shows that the tangential acceleration of the ball is not constant, and we cannot make further progress using kinematics equations. In other words, we cannot find the speed using kinematics. So, our use of energy conservation was the right way to approach the problem.

The frictional force is $f = \frac{2}{5}ma = \frac{2}{5}m \times \frac{5g\sin\theta}{7} = \frac{2mg\sin\theta}{7}$. So, the question now is how we could use energy conservation when a frictional force was present. The answer is again the fact that the ball was rolling without slipping. The point of contact of the ball with the sphere was instantaneously at rest: the surface of the ball does not move relative to the surface of the sphere, so the frictional force does zero work thus allowing the use of energy conservation.

These results apply to the case where the ball was given an infinitesimal initial speed to start moving.

An interesting variant of the problem is to ask what initial horizontal velocity should be given to the ball so that it immediately leaves the sphere.

It will very helpful, as we will see, to replace the factor of $\frac{2}{5}$ in the moment of inertia of the sphere by *k*. I.e. we will write the moment of inertial of the sphere as $I = kmr^2$.

If we give the ball an initial horizontal velocity *u* then the set of equations:

$$mg(R+r)(1-\cos\theta) = \frac{1}{2}mv^2 + \frac{1}{2} \times \frac{2}{5}mr^2\frac{v^2}{r^2}$$
$$2g(R+r)(1-\cos\theta) = v^2 + \frac{2}{5}v^2 = \frac{7}{5}v^2$$
$$v^2 = \frac{10}{7}g(R+r)(1-\cos\theta)$$

change to

$$\frac{1}{2}mu^2+\frac{1}{2}\times kmr^2\frac{u^2}{r^2}+mg(R+r)(1-\cos\theta)=\frac{1}{2}mv^2+\frac{1}{2}\times kmr^2\frac{v^2}{r^2}$$
$$(1+k)u^2+2g(R+r)(1-\cos\theta)=(1+k)v^2$$
$$v^2=u^2+\frac{2g(R+r)(1-\cos\theta)}{1+k}$$

Therefore, the set:

$$mg\cos\theta=m\frac{v^2}{R+r}$$
$$=\frac{10}{7}mg(1-\cos\theta)$$

changes to

$$g\cos\theta=\frac{u^2}{R+r}+\frac{2g(1-\cos\theta)}{1+k}$$

so that

$$(1+k)(R+r)g\cos\theta=(1+k)u^2+2g(R+r)(1-\cos\theta)$$
$$g(R+r)\cos\theta(1+k+2)=(1+k)u^2+2g(R+r)$$
$$\cos\theta=\frac{(1+k)u^2+2g(R+r)}{(3+k)g(R+r)}$$

We can check this result against our previous results. For zero initial speed and for a point particle, u = 0 and k = 0 (no moment of inertia) and so

$$\cos\theta_{\text{particle}}=\frac{0+2g(R+0)}{(3+0)g(R+0)}=\frac{2}{3}\text{ as before.}$$

For a sphere with zero initial speed, u = 0 and $k=\frac{2}{5}$,

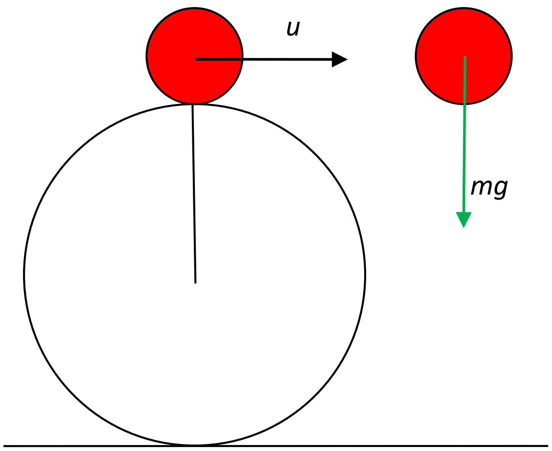
$$\cos\theta_{\text{sphere}}=\frac{0+2g(R+r)}{(3+\frac{2}{5})g(R+r)}=\frac{10}{17}\text{ as before.}$$

Back to the problem. We want to ball to leave the sphere immediately i.e. at $\theta=0\Rightarrow\cos\theta=1$ and so

$$(1+k)u^2+2g(R+r)=(3+k)g(R+r)$$
$$(1+k)u^2=(1+k)g(R+r)$$
$$u^2=g(R+r)$$

The amazing thing here is that all dependence on k has disappeared. If we had started with a point particle it would have no moment of inertia and so $k = 0$. But the result for the initial speed u would be the same as that of a sphere!

These results can be understood conceptually as follows:



For the ball to leave immediately the normal force will be zero. In the vertical direction the only force is then the weight vertically down. The centripetal acceleration is then g and so the speed required is

$a_c = g = \frac{u^2}{R}$ for the point particle and $a_c = g = \frac{u^2}{R+r}$ for the sphere! These equations give the required speed u.

This much simpler way of deriving u also shows why the result is the same for a point particle and a sphere: the centripetal acceleration being g is all that is required and so rotation and moments of inertia play no role.